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A potential problem arising from the strip-punch problem in elasticity

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Abstract. Three-dimensional contact problems in the classical theory of linear elasticity can often be regarded as mixed boundary-value problems of potential theory. In this paper we examine the problem where contact between the indenting object (called a punch) and the elastic medium is maintained over an infinite strip. It is assumed that a rigid frictionless punch with a known profile has indented a homogeneous, isotropic and linearly elastic half-space. Applying the theory of Mathieu functions, an analytic solution of Laplace's equation is obtained through separation of variables in the elliptic cylinder coordinate system. Finally three examples are discussed where in each case the normal component of stress under the punch is numerically evaluated.

1. Introduction

Consider an elastic medium occupying the infinite half-space $z \geqslant 0$. A punch (or indentation) problem, in the theory of elasticity, is a problem where a body (called a punch) is pressed against the elastic medium under the action of a normal force, as a result of which certain displacements and stresses are created within and on the boundary of the medium.

Let S be the region of contact, that is, the part of the boundary of the elastic half-space consisting of those points which after deformation are in contact with the displaced surface of the base of the punch, and let $\bar{S}$ be the region of the boundary of the half-space outside S.

Throughout this paper we shall assume the following:

(a) the elastic medium is linearly elastic, homogeneous and isotropic,
(b) the punch is a perfectly rigid body,
(c) there is no friction between the punch and the surface of the elastic medium,
(d) the normal component of stress is zero on $\bar{S}$,
(e) there is complete contact between the base of the punch and the elastic medium.

The problem of static equilibrium, and in particular the problem of determining the state of stress in an elastic half-space where part of its boundary is subjected to a normal force Q, can be reduced to a mixed boundary-value problem in potential theory. The displacement and the state of stress of an elastic medium under normal loading, where the normal component of stress, τ_{zz}, is prescribed on part of the boundary, the normal component of displacement, w, is given on another part of the boundary, and shear stresses are absent, can be determined when we have found a function $\Psi(x, y, z)$ which is harmonic everywhere except on the region S of loading and vanishes at infinity with the following behavior:

$\Psi \sim Q/R$, where

$$Q = \iint_S p(x, y)\, dx\, dy, \quad R = (x^2 + y^2 + z^2)^{1/2},$$

and $p(x, y)$ is the normal pressure applied to the punch. Then the Papkovich–Neuber solution ([6], Ch. 1, Sec. 1.10)

$$2\mu \mathbf{d} = 4(1 - \nu)\boldsymbol{\Psi} - \nabla\{(\mathbf{r} \cdot \boldsymbol{\Psi}) + \Phi\} \tag{1.1}$$

to the problem of elastic equilibrium can be used to arrive at the required stress and displacement components. In (1.1), $\mathbf{d}(u, v, w)$ is the displacement vector, $\mathbf{r}$ is the position vector of a field point, $\boldsymbol{\Psi}$ and Φ are a pair of vector and scalar functions, respectively, which in the absence of body forces satisfy $\nabla^2\boldsymbol{\Psi} = \mathbf{0}$, and $\nabla^2\Phi = 0$; ν is Poisson's ratio and μ is the shear modulus (both constants).

Following the notation of Gladwell [6], if $\boldsymbol{\Psi}$ is chosen so that $\boldsymbol{\Psi} = (0, 0, \Psi)$ then the components of displacement $\mathbf{d} = (u, v, w)$ are given by

$$2\mu u = -z\frac{\partial \Psi}{\partial x} - \frac{\partial \Phi}{\partial x}, \tag{1.2a}$$

$$2\mu v = -z\frac{\partial \Psi}{\partial y} - \frac{\partial \Phi}{\partial y}, \tag{1.2b}$$

and

$$2\mu w = 4(1 - \nu)\Psi - \left(z\frac{\partial \Psi}{\partial z} + \Psi + \frac{\partial \Phi}{\partial z}\right). \tag{1.2c}$$

The corresponding components of the stress tensor are given by

$$\tau_{xz} = (1 - 2\nu)\frac{\partial \Psi}{\partial x} - z\frac{\partial^2 \Psi}{\partial x \partial z} - \frac{\partial^2 \Phi}{\partial x \partial z}, \tag{1.3a}$$

$$\tau_{yz} = (1 - 2\nu)\frac{\partial \Psi}{\partial y} - z\frac{\partial^2 \Psi}{\partial y \partial z} - \frac{\partial^2 \Phi}{\partial y \partial z}, \tag{1.3b}$$

$$\tau_{zz} = 2(1 - \nu)\frac{\partial \Psi}{\partial z} - z\frac{\partial^2 \Psi}{\partial z^2} - \frac{\partial^2 \Phi}{\partial z^2}. \tag{1.3c}$$

For zero shearing stress on $z = 0$ we have $\tau_{xz}(x, y, 0) = \tau_{yz}(x, y, 0) = 0$ for all x and y, and

$$(1 - 2\nu)\Psi = \frac{\partial \Phi}{\partial z}. \tag{1.4}$$

Consequently the normal component of stress is given by

$$\tau_{zz} = \frac{\partial \Psi}{\partial z} - z\frac{\partial^2 \Psi}{\partial z^2}.$$

Thus we obtain two quantities of particular interest, $w(x, y, 0)$ and $\tau_{zz}(x, y, 0)$, i.e., the normal component of displacement on $\bar{S}$ and the normal component of stress on S, respectively

$$w(x, y, 0) = \frac{(1 - \nu)}{\mu}\Psi(x, y, 0),$$

$$\tau_{zz}(x, y, 0) = \left.\frac{\partial \Psi}{\partial z}\right|_{z=0}.$$

It can be shown [6] that this special case of the Papkovich–Neuber solution is satisfied by a representation of Ψ in the form

$$\Psi(x, y, z) = \frac{1}{2\pi}\iint_S \frac{p(x', y')}{R_1}\,\mathrm{d}x'\,\mathrm{d}y' \tag{1.5}$$

where

$$R_1 = [(x - x')^2 + (y - y')^2 + z^2]^{1/2}$$

is the distance from the point (x, y, z) of the elastic medium to the point $(x', y', 0)$ of the surface and $\tau_{zz}(x, y, 0) = -p(x, y)$.

Furthermore, the function Φ, which is required in the derivation of the components of displacement u and v (1.2a,b) can be found from (1.4) and (1.5):

$$\Phi(x, y, z) = \frac{(1 - 2\nu)}{2\pi}\iint_S \ln\,(z + R)p(x', y')\,\mathrm{d}x'\,\mathrm{d}y'.$$

It is important to note that the potential problem, when solved, gives a solution of the punch problem only if $\tau_{zz}(x, y, 0) < 0$ for all (x, y) in S. This is due to the requirement that there should be complete contact between the base of the punch and the medium.

The method of employing an appropriate coordinate system and solving Laplace's equation in that system has been used by Luré [9] and Shail [11]. Luré solves several contact problems where the contact region S is assumed to be circular and Shail solves the problem where S is elliptic. Here, however, we shall provide a solution for the case where S is an infinite strip.

2. Formulation of the general boundary-value problem

Let the contact region, S, be defined in terms of the Cartesian coordinates (x, y, z), by $-\infty < x < \infty$, $|y| < f$ and $z = 0$. A rigid frictionless punch is applied to the region S, its profile being given by a function $K(x, y)$. The boundary conditions can be stated as:

$$w(x, y, 0) = K(x, y) \quad \text{on } S,$$

$$\tau_{zz}(x, y, 0) = 0 \quad \text{on } \bar{S}.$$

Hence we seek a solution to the boundary-value problem for the harmonic function Ψ, where for the elastic medium we have

(i) $\nabla^2\Psi = 0$ for $z > 0$,
(ii) $\Psi \to 0$ as $R \to \infty$, ($R = (x^2 + y^2 + z^2)^{1/2}$), in $z \geqslant 0$,
(iii) $\partial\Psi/\partial z = 0$ on $\bar{S}$,
(iv) $\{(1 - \nu)/\mu\}\ \Psi(x, y, 0) = K(x, y)$ on S.

The function $K(x, y)$ can always be expressed as the sum of four functions each having symmetry or antisymmetry about one of the axes $x = 0, y = 0$, and because of linearity we can superpose solutions corresponding to these four functions. To simplify the analysis, therefore, we shall assume that

(v) $K(x, y)$ is symmetric about $y = 0$,
(vi) $K(x, y)$ is symmetric about $x = 0$.

We are now going to transform to elliptic cylinder coordinates. This transformation will change the boundary values accordingly so that they will no longer be mixed [3]. Let the elliptic cylinder coordinates of a point be given by the variables (x, ξ, η) which are related to the Cartesian coordinates by

$$x = x, \quad y = f\cosh\xi\cos\eta, \quad z = f\sinh\xi\sin\eta, \tag{2.1}$$

where $-\pi < \eta \leqslant \pi$, and $\xi \geqslant 0$. The surfaces corresponding to ξ = constant consist of a family of confocal elliptic cylinders; that for which $\xi = \xi_0$ is such that its section by the plane $x = 0$ is an ellipse with foci $(0, \pm f, 0)$, eccentricity sech ξ_0. For $\xi = 0$ we get the degenerate surface consisting of an infinite strip in the x,y-plane of finite width $2f$. This is merely the case of an elliptic cylinder of eccentricity 1 with zero minor axis and finite major axis $2f$. The surfaces corresponding to η = constant are portions of confocal hyperbolic cylinders which are normal to the surfaces ξ = constant.

3. The general solution of the boundary-value problem

We now transform to the elliptic cylinder coordinate system where η will be restricted to $0 \leqslant \eta \leqslant \pi$ since we are only concerned with the half-space occupied by the elastic medium. In terms of (x, ξ, η), Laplace's equation $\nabla^2\Psi = 0$ is given by

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{2}{f^2(\cosh 2\xi - \cos 2\eta)}\left(\frac{\partial^2\Psi}{\partial\xi^2} + \frac{\partial^2\Psi}{\partial\eta^2}\right) = 0. \tag{3.1}$$

Let $\Psi = X(x)F(\xi)G(\eta)$, then

$$\frac{X''}{X} + \frac{2}{f^2(\cosh 2\xi - \cos 2\eta)}\left(\frac{F''}{F} + \frac{G''}{G}\right) = 0.$$

The separated equations are:

$$X'' = \alpha X, \tag{3.2a}$$

$$F'' + (\tfrac{1}{2}\alpha f^2 \cosh 2\xi - \beta)F = 0, \tag{3.2b}$$

$$G'' + (\beta - \tfrac{1}{2}\alpha f^2 \cos 2\eta)G = 0, \tag{3.2c}$$

where α and β are separation constants.

Equation (3.2c) is Mathieu's equation and equation (3.2b) is the modified Mathieu equation. In making use of Mathieu functions we shall follow the notation of McLachlan [10].

In view of the change of coordinates, conditions (i) to (vi) in Section 2 are to be replaced by

(i)′ equation (3.1) holds for $\xi \in (0, \infty)$, $\eta \in (0, \pi)$ and $x \in (-\infty, \infty)$,
(ii)′ $\Psi \to 0$ as $|x| \to \infty$ or $\xi \to \infty$, for $\eta \in [0, \pi]$,
(iii)′ since

$$\frac{\partial}{\partial z} = \frac{\cosh \xi \sin \eta}{f(\sinh^2 \xi + \sin^2 \eta)}\frac{\partial}{\partial \xi} + \frac{\sinh \xi \cos \eta}{f(\sinh^2 \xi + \sin^2 \eta)}\frac{\partial}{\partial \eta}$$

and $\bar{S}$ is the region where $\xi > 0$, $\eta = 0$, or π, then $\partial\Psi/\partial z = 0$ on $\bar{S}$ is equivalent to

$$\frac{1}{f \sinh \xi}\frac{\partial \Psi}{\partial \eta} = 0 \quad \text{at} \quad \eta = 0 \quad \text{and} \quad \eta = \pi$$

where $\xi \in (0, \infty)$ and $x \in (-\infty, \infty)$,
(iv)′ $\Psi(x, \eta, 0) = H(x, \eta)$, where $\eta \in (0, \pi)$, $x \in (-\infty, \infty)$ and $H(x, \eta) \stackrel{\text{d.}}{=} \{\mu/(1 - \nu)\}K(x, y)$,
(v)′ $H(x, \eta)$ is symmetric about $\eta = \pi/2$,
(vi)′ $H(x, \eta)$ is symmetric about $x = 0$.

For X in (3.2a) to be finite, α must be negative. Let $\alpha = -k^2$ so

$$X = A \cos kx + B \sin kx. \tag{3.3}$$

Since the solution Ψ is assumed to have the form $\Psi = X(x)F(\xi)G(\eta)$, we require $X(x)$ and $G(\eta)$ to have properties corresponding to the symmetries of $H(x, \eta)$ given by (v)′ and (vi)′. In the first place (vi)′ implies that $X = A \cos kx$. Also since $\alpha = -k^2$, if we let $k^2 f^2 = 4h^2$, then equation (3.2c) becomes

$$G'' + (\beta + 2h^2 \cos 2\eta)G = 0. \tag{3.4}$$

This is Mathieu's equation, in which the parameter usually written as q is negative. This point is particularly relevant when we use (as we shall later) the so-called functions of the third kind.

There are four types of basically periodic solutions of (3.4) (i.e., of period π or 2π) called Mathieu functions of integral order of the first kind. Two of these are even while the other two are odd, and they are expressed by the following expansions [8]:

$$\mathrm{ce}_{2n}(\eta, -h^2) = \sum_{r=0}^{\infty} A_{2r}^{(2n)}(-h^2) \cos 2r\eta, \tag{3.5a}$$

$$\mathrm{ce}_{2n+1}(\eta, -h^2) = \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(-h^2) \cos (2r+1)\eta, \tag{3.5b}$$

$$\mathrm{se}_{2n+1}(\eta, -h^2) = \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)}(-h^2) \sin (2r+1)\eta, \tag{3.5c}$$

$$\mathrm{se}_{2n+2}(\eta, -h) = \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)}(-h^2) \sin (2r+2)\eta. \tag{3.5d}$$

It should be noted here that the above four functions are possible solutions of equation (3.4) provided β (which is dependent on h^2) is one of the countably infinite real eigenvalues of (3.4). The corresponding eigenvalues for the expressions (3.5a, b, c, d) are denoted respectively by $a_{2n}(-h^2)$, $a_{2n+1}(-h^2)$, $b_{2n+1}(-h^2)$ and $b_{2n+2}(-h^2)$, where n is a positive integer or zero. We also know that in this case (i.e., when the equation has as solution a periodic Mathieu function of one of the four types above) the second solution is not periodic ([1], Sec. 2.4.1).

From condition (iii)′ we have $G'(\pi) = G'(0) = 0$, which implies that G is a Matheiu function of the first kind (i.e., of period π or 2π)([1], Sec. 2.1.1), and $G'(0) = 0$ implies that G must be $\mathrm{ce}_{2n}(\eta, -h^2)$ or $\mathrm{ce}_{2n+1}(\eta, -h^2)$. Finally from condition (v)′, $G(\eta) = \mathrm{ce}_{2n}(\eta, -h^2)$ and hence we can let $\beta = a_{2n}(-h^2)$.

Next, equation (3.2b) implies that

$$F'' + (-2h^2 \cosh 2\xi - a_{2n})F = 0, \tag{3.6}$$

whose four solutions with period πi and 2πi are given by McLachlan [10]:

$$\mathrm{Ce}_{2n}(\xi, -h^2) \overset{\text{d.}}{=} \mathrm{ce}_{2n}(\mathrm{i}\xi, -h^2) = \sum_{r=0}^{\infty} A_{2r}^{(2n)}(-h^2) \cosh 2r\xi,$$

$$\mathrm{Ce}_{2n+1}(\xi, -h^2) \overset{\text{d.}}{=} \mathrm{ce}_{2n+1}(\mathrm{i}\xi, -h^2) = \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}(-h^2) \cosh (2r+1)\xi,$$

$$\mathrm{Se}_{2n+1}(\xi, -h^2) \overset{\text{d.}}{=} (-\mathrm{i})\mathrm{se}_{2n+1}(\mathrm{i}\xi, -h^2) = \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)}(-h^2) \sinh (2r+1)\xi,$$

$$\mathrm{Se}_{2n+2}(\xi, -h^2) \overset{\text{d.}}{=} (-\mathrm{i})\mathrm{se}_{2n+2}(\mathrm{i}z, -h^2) = \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)}(-h^2) \sinh (2r+2)\xi.$$

We have, of course, many possible ways of choosing a second solution independent of $\mathrm{Ce}_m(\xi, -h^2)$ (or $\mathrm{Se}_m(\xi, -h^2)$).

One such solution is denoted by $\mathrm{Fek}_m(\xi, -h^2)$ (or $\mathrm{Gek}_m(\xi, -h^2)$ respectively). These functions are expressible in infinite series of the K-Bessel functions (modified Bessel functions of the third kind K_ν). For example [10],

$$\mathrm{Fek}_{2n}(\xi, -h^2) = (-1)^n \frac{\mathrm{ce}_{2n}(0, h^2)}{\pi A_0^{(2n)}(h^2)} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)}(h^2) K_{2r}(2h \cosh \xi).$$

The usefulness of $\mathrm{Fek}_{2n}(\xi, -h^2)$ lies in its asymptotic behavior as $\xi \to \infty$. The following asymptotic forms are given by McLachlan ([10], Sec. 11.12):

$$\mathrm{Ce}_{2n}(\xi, -h^2) \sim \frac{P'_{2n}(h)}{(2\pi)^{1/2}} v^{-1/2} \mathrm{e}^{v}, \quad \text{as } \xi \to \infty,$$

$$\mathrm{Fek}_{2n}(\xi, -h^2) \sim \frac{P'_{2n}(h)}{(2\pi)^{1/2}} v^{-1/2} \mathrm{e}^{-v}, \quad \text{as } \xi \to \infty,$$

where

$$P'_{2n}(h) = (-1)^n \mathrm{ce}_{2n}(0, h^2) \mathrm{ce}_{2n}\left(\frac{\pi}{2}, h^2\right) \Big/ A_0^{(2n)}(h^2),$$

and $v = h\mathrm{e}^{\xi}$.

Now we can choose $\mathrm{Ce}_{2n}(\xi, -h^2)$ and $\mathrm{Fek}_{2n}(\xi, -h^2)$ as a pair of linearly independent solutions of (3.6). However condition (ii)′ requires that the general solution tends to zero as $\xi \to \infty$, and since $\mathrm{Fek}_{2n}(\xi, -h^2)$ is the only one which shows this behavior, we must exclude the solution $\mathrm{Ce}_{2n}(\xi, -h^2)$. Hence a separated solution is of the form

$$\begin{aligned} \Psi_n &\stackrel{\text{d.}}{=} \Psi_n(x, \xi, \eta, h) \\ &= B_n(h) \cos \frac{2hx}{f} \mathrm{ce}_{2n}(\eta, h^2) \, \mathrm{Fek}_{2n}(\xi, -h^2), \end{aligned}$$

where n is an arbitrary non-negative integer, h is an arbitrary non-negative parameter, and $B_n(h)$ an arbitrary constant, written in this way since n is an integer-valued parameter while h is continuous.

The above solution is, however, a single separated solution and cannot be expected to satisfy the remaining boundary condition (iv)′. Since the parameter n is discrete whereas the parameter h is continuous and can take any non-negative value it is natural to superpose solutions by summing over n from zero to $+\infty$ and integrating with respect to h from zero to $+\infty$. The coefficient $B_n(h)$ can then be determined if we let the general solution satisfy condition (iv)′.

A general formal solution is thus given by

$$\Psi = \int_0^\infty \sum_{n=0}^{\infty} B_n(h) \cos \frac{2hx}{f} \mathrm{ce}_{2n}(\eta, -h^2) \, \mathrm{Fek}_{2n}(\xi, -h^2) \, \mathrm{d}h. \tag{3.7}$$

Ignoring questions of convergence for the moment, and proceeding formally with the solution, (iv)′ implies

$$H(x, \eta) = \int_0^\infty \sum_{n=0}^\infty C_n(h)\, \mathrm{ce}_{2n}(\eta, -h^2) \cos \frac{2hx}{f}\, \mathrm{d}h, \tag{3.8}$$

where

$$C_n(h) = B_n(h)\, \mathrm{Fek}_{2n}\,(0, -h^2).$$

Writing (3.8) as

$$H(x, \eta) = \int_0^\infty g(h, \eta) \cos \frac{2hx}{f}\, \mathrm{d}h,$$

where

$$g(h, \eta) = \sum_{n=0}^\infty C_n(h)\, \mathrm{ce}_{2n}\,(\eta, -h^2),$$

and using the Fourier cosine transform formulas, we get

$$g(h, \eta) = \frac{4}{f\pi} \int_0^\infty H(x, \eta) \cos \frac{2hx}{f}\, \mathrm{d}x$$

(provided the integral exists).

Multiplying both sides of the above equation by $\mathrm{ce}_{2m}(\eta, -h^2)$ and integrating with respect to η from zero to π, we get:

$$\int_0^\pi \sum_{n=0}^\infty C_n(h)\, \mathrm{ce}_{2n}(\eta, -h^2)\mathrm{ce}_{2m}(\eta, -h^2)\, \mathrm{d}\eta$$

$$= \int_0^\pi \frac{4}{f\pi} \mathrm{ce}_{2m}(\eta, -h^2) \int_0^\infty H(x, \eta) \cos \frac{2hx}{f}\, \mathrm{d}x\, \mathrm{d}\eta. \tag{3.9}$$

Still proceeding formally, we interchange the order of integration and summation on the left-hand side of (3.9) and use the orthogonality of Mathieu functions ([10], Sections 2.19, 2.21) to deduce that the left-hand side is equal to $(\pi/2)C_m(h)$. Therefore

$$B_n(h) = \frac{8}{f\pi^2\, \mathrm{Fek}_{2n}(0, -h^2)} \int_0^\pi \int_0^\infty H(x, \eta)\, \mathrm{ce}_{2n}\,(\eta, -h^2) \cos \frac{2hx}{f}\, \mathrm{d}x\, \mathrm{d}\eta. \tag{3.10}$$

Having evaluated $B_n(h)$, the solution Ψ of the problem is then given by (3.7). From the equations of elastostatics (1.2a, b, c) and (1.3a, b, c) the corresponding displacements and stresses can be obtained. In particular,

$$w(x, y, 0) = \frac{(1 - \nu)}{\mu} \int_0^\infty \sum_{n=0}^\infty B_n(h) \cos \frac{2hx}{f}\, \mathrm{ce}_{2n}(0, -h^2)\mathrm{Fek}_{2n}\left(\cosh^{-1} \frac{|y|}{f}, -h^2\right) \mathrm{d}h. \tag{3.11}$$

Moreover, using condition (iii)′ and (3.7), the normal component of stress under the punch (i.e., on S) can be expressed by

$$\tau_{zz}(x, y, 0) = (f^2 - y^2)^{-1/2} \int_0^\infty \sum_{n=0}^{\infty} B_n(h) \cos \frac{2hx}{f} \times \mathrm{ce}_{2n}\left(\cos^{-1}\frac{y}{f}, -h^2\right) \mathrm{Fek}'_{2n}(0, -h^2)\, dh, \tag{3.12}$$

where $|y| < f$.

We observe that the above function has singularities at $y = \pm f$, i.e., at the edge of the contact region. In punch problems where complete contact is assumed, one expects to find stress singularities of the square-root type at the edge of the contact region.

4. Validity of the formal solution

The validity of the formal solution thus obtained depends of course on the behavior of the prescribed function $H(x, \eta)$.

For a given profile $H(x, \eta)$, one can examine the integral (3.10) for convergence and for differentiability with respect to x, ξ, and η, then proceed to a corresponding investigation of the expression for Ψ in (3.7).

More generally, the formal solution may be shown to be valid by specifying a set of sufficient conditions on $H(x, \eta)$. In what follows we shall assume that $H(x, \eta)$ satisfies a set of eight conditions which are clearly reasonable from a physical point of view.

Let us begin by imposing the following conditions on $H(x, \eta)$:

(c.1) There exists a function $A_0(x)$, such that $|H(x, \eta)| < A_0(x)$ for all $\eta \in [0, \pi]$ and $A_0(x) \in L[0, \infty)$.
(c.2) $H(x, \eta)$ is a continuous function of both x and η for all $x \in [0, \infty)$ and all $\eta \in [0, \pi]$.
(c.3) As a function of η, $H(x, \eta)$ is four times continuously differentiable (i.e., partially with respect to η) for all $\eta \in [0, \pi]$ and all $x \in [0, \infty)$.
(c.4) For $i = 1, 2, 3, 4$, there exist functions $A_i(x)$ such that $|\partial^i H(x, \eta)/\partial\eta^i| < A_i(x)$, for all $\eta \in [0, \pi]$, and $A_i(x) \in L[0, \infty)$.
(c.5) $\partial H(x, \eta)/\partial x$ is a continuous function of both x and η for all $x \in [0, \infty)$ and all $\eta \in [0, \pi]$.
(c.6) For each $x \in [0, \infty)$, $\partial^j H(x, \eta)/\partial\eta^j = 0$ at $\eta = 0$, and $\eta = \pi$, for $j = 1, 3$.

Before stating the remaining conditions, we define

$$M_0(h) = \max_{0 \leqslant \eta \leqslant \pi} |T|$$

and for $i = 1, 2, 3, 4$,

$$M_i(h) = \max_{0 \leqslant \eta \leqslant \pi} \left|\frac{\partial^i T}{\partial \eta^i}\right|,$$

where

$$T = T(h, \eta) \stackrel{\text{d.}}{=} \int_0^\infty H(x, \eta) \cos \frac{2hx}{f} \, \mathrm{d}x.$$

(The existence of these expressions is ensured by conditions (c.1) and (c.4) above.)

(c.7) Then for integers i and m, where $0 \leqslant i \leqslant 4$ and $0 \leqslant m \leqslant 8$, $\int_0^\infty h^m M_i(h) \, \mathrm{d}h < \infty$.

(c.8) There exists a constant, K, such that for $h \in [0, \infty)$ and for integers i and m, where $0 \leqslant i \leqslant 4$ and $0 \leqslant m \leqslant 8$, $h^m M_i(h) \leqslant K$.

In addition it should be kept in mind that $H(x, \eta)$ is assumed to be an even function of x and $(\eta - \pi/2)$.

Let

$$T(h, \eta) \stackrel{\text{d.}}{=} \int_0^\infty H(x, \eta) \cos \frac{2hx}{f} \, \mathrm{d}x, \tag{4.1}$$

then (c.1) implies the existence of $T(h, \eta)$ for all $h \geqslant 0$ and $\eta \in [0, \pi]$.

Next we expand $T(h, \eta)$ as a Mathieu function series:

$$T(h, \eta) = \sum_{n=0}^{\infty} D_n(h) \, \mathrm{ce}_{2n}(\eta, -h^2). \tag{4.2}$$

From the general theory of Sturm-Liouville expansions ([7], Sec. 11.5) we know that if, for any fixed real h, $T(h, \eta)$ is a continuous function of η, where η belongs to some finite interval, then the Fourier series and the Mathieu-function series expansions of $T(h, \eta)$ are equiconvergent (i.e., the two series will converge under exactly the same conditions) on the same finite interval. Now (4.1), (c.1) and (c.2) together imply that $T(h, \eta)$ is a continuous function of η for each fixed $h \geqslant 0$. Furthermore (c.3) and (c.4) imply that $\partial T(h, \eta)/\partial \eta$ is also a continuous function of η for all $\eta \in [0, \pi]$ and each fixed $h \geqslant 0$. Hence for each $h \geqslant 0$, the infinite series (4.2) converges uniformly in η to $T(h, \eta)$. The coefficients $D_n(h)$ are found by the usual technique (analogous to that for the Fourier series coefficients) as follows:

Multiplying both sides of (4.2) by $\mathrm{ce}_{2m}(\eta, -h^2)$, integrating with respect to η from 0 to π and applying orthogonality properties of Mathieu functions, we get

$$\int_0^\pi \mathrm{ce}_{2n}(\eta, -h^2) T(h, \eta) \, \mathrm{d}\eta = \frac{\pi}{2} D_n(h) \tag{4.3}$$

(term by term integration of $\Sigma_{n=0}^{\infty} D_n(h) \mathrm{ce}_{2n}(\eta, -h^2) \mathrm{ce}_{2m}(\eta, -h^2)$ is permitted since this is a uniformly convergent series of continuous functions of η, for each fixed $h \geqslant 0$).

Now let

$$B_n(h) = \frac{4 D_n(h)}{\pi f \, \mathrm{Fek}_{2n}(0, -h^2)} \tag{4.4}$$

and note that $\mathrm{Fek}_{2n}(0, -h^2) \neq 0$ for any $n = 0, 1, 2, \ldots$, and any $h \geqslant 0$ ([4], Appendix A).

So

$$\frac{4}{f\pi} T(h, \eta) = \sum_{n=0}^{\infty} B_n(h) \operatorname{Fek}_{2n}(0, -h^2) \operatorname{ce}_{2n}(\eta, -h^2)$$

and

$$\int_0^\infty \cos\frac{2hx}{f} \sum_{n=0}^{\infty} B_n(h) \operatorname{Fek}_{2n}(0, -h^2) \operatorname{ce}_{2n}(\eta, -h^2)\, dh$$

$$= \frac{4}{f\pi}\int_0^\infty \cos\frac{2hx}{f}\, T(h, \eta)\, dh$$

$$= \frac{4}{f\pi}\int_0^\infty \int_0^\infty \cos\frac{2hx}{f}\, H(x', \eta) \cos\frac{2hx'}{f}\, dx'\, dh$$

$$= H(x, \eta).$$

The validity of the last step, which states that $H(x, \eta)$ is equal to the inverse cosine transform of its transform is ensured by (c.1), (c.2) and (c.5). Hence expression (3.8) is justified, where $C_n(h) = \{4/(\pi f)\}D_n(h)$, and $B_n(h)$ is given by (3.10). To show that the function $\Psi(x, \xi, \eta)$, given by (3.7), is a continuous function of x, ξ and η, we start by rewriting Ψ as

$$\Psi(x, \xi, \eta) = \frac{4}{\pi f}\int_0^\infty \cos\frac{2hx}{f} \sum_{n=0}^{\infty} D_n(h) \operatorname{ce}_{2n}(\eta, -h^2) \frac{\operatorname{Fek}_{2n}(\xi, -h^2)}{\operatorname{Fek}_{2n}(0, -h^2)}\, dh \tag{4.5}$$

where $D_n(h)$ is given by (4.3). For $n = 0, 1, 2, \ldots, h \geqslant 0$ and $\xi \geqslant 0$,

$$0 < \frac{\operatorname{Fek}_{2n}(\xi, -h^2)}{\operatorname{Fek}_{2n}(0, -h^2)} \leqslant 1,$$

$$|\operatorname{ce}_{2n}(\eta, -h^2)| \leqslant \gamma_0 + \gamma_1 h + \gamma_2 h^2$$

where γ_0, γ_1, and γ_2 are positive constants, and

$$\left|\frac{\pi}{2}(a_{2n} + 2h^2 + 1)D_n(h)\right| \leqslant \frac{\pi}{\sqrt{2}}(1 + 4h^2)M_0(h) + \frac{\pi}{\sqrt{2}} M_2(h)$$

([4], Appendices B, C, D) where $M_0(h)$ and $M_2(h)$ satisfy condition (c.7). From this inequality it follows that

$$|D_n(h)| \leqslant \frac{2[(1 + 4h^2)M_0(h) + M_2(h)]}{4n^2 + 1}.$$

Hence

$$\left| D_n(h)\, \mathrm{ce}_{2n}\,(\eta,\ -h^2)\, \frac{\mathrm{Fek}_{2n}(\xi,\ -h^2)}{\mathrm{Fek}_{2n}(0,\ -h^2)} \right|$$

$$\leqslant \left[M_0(h) \sum_{i=0}^{4} \alpha_i h^i + M_2(h) \sum_{j=0}^{2} \beta_j h^j \right] (4n^2 + 1)^{-1}$$

where α_i and β_j are positive constants.

Using condition (c.8) and Weierstrass's M-test we deduce that

$$\sum_{n=0}^{\infty} D_n(h) \mathrm{ce}_{2n}(\eta,\ -h^2)\, \frac{\mathrm{Fek}_{2n}(\xi,\ -h^2)}{\mathrm{Fek}_{2n}(0,\ -h^2)}$$

is uniformly convergent with repect to η, ξ and h. Furthermore, from the general theory of Mathieu equations, $\mathrm{ce}_{2n}(\eta,\ -h^2)$ is continuous in η and h, $\mathrm{Fek}_{2n}(\xi,\ -h^2)$ is continuous in ξ and h, and from (4.1), (4.3), (c.1) and (c.2) $D_n(h)$ is also continuous. Hence the function represented by the above infinite series is continuous in η, ξ and h.

Next let

$$\Psi_N(x,\ \xi,\ \eta) \ = \ \frac{4}{\pi f} \int_0^N \cos \frac{2hx}{f} \sum_{n=0}^{\infty} D_n(h)\, \mathrm{ce}_{2n}\,(\eta,\ -h^2)\, \frac{\mathrm{Fek}_{2n}(\xi,\ -h^2)}{\mathrm{Fek}_{2n}(0,\ -h^2)}\, \mathrm{d}h,$$

then for each $N = 1, 2, \ldots,\ \Psi_N$ is a continuous function of x, ξ and η and by condition (c.7), the sequence of functions Ψ_N converges uniformly to Ψ. Hence $\Psi(x,\ \xi,\ \eta)$ is a continuous function of x, ξ, and η.

Differentiating the integrand in the expression for Ψ, (4.5), twice partially with respect to x only introduces factors h and h^2 in the integrand. In either case the uniform convergence of the integrand is ensured by condition (c.7) and the continuity of the integrand, as a function of x, ξ, η and h, is not affected.

From the inequalities for the derivatives of ce_{2n} and Fek_{2n}([4], Appendices) it follows that Ψ is twice partially differentiable with respect to ξ and η. Furthermore one can use the Riemann–Lebesgue lemma to show that for each η in $[0, \pi]$, $\Psi \to 0$ as $|x| \to \infty$ or $\xi \to \infty$.

Therefore $\Psi(x,\ \xi,\ \eta)$ represented by (3.7) is continuous and satisfies Laplace's equation as well as the boundary conditions of the stated boundary-value problem, provided the function $H(x,\ \eta)$ satisfies the conditions (c.1) to (c.8).

5. Examples

Example 1

We consider first a situation in which the depth of the punch profile varies only longitudinally. Let $H(x,\ \eta) = \delta l^2/(l^2 + x^2)$ where δ and l are parameters having the dimension of length, δ measuring the maximum depth of the punch which occurs at the origin.

Then

$$T(h, \eta) = \delta \int_0^\infty \cos\left(\frac{2hx}{f}\right) \frac{l^2}{l^2 + x^2}\, \mathrm{d}x = \frac{2l\delta}{f} \exp\left(-\frac{2lh}{f}\right).$$

Also

$$M_0(h) = \frac{2l\delta}{f} \exp\left(-\frac{2lh}{f}\right)$$

and for $i = 1, 2, 3, 4$, $M_i(h) = 0$, so conditions (c.1) to (c.8) of Section 4 are satisfied. Moreover

$$D_n(h) = \frac{4l\delta}{\pi f} \exp\left(-\frac{2lh}{f}\right) \int_0^\pi \mathrm{ce}_{2n}(\eta, -h^2)\, \mathrm{d}\eta \tag{5.1}$$

and

$$\int_0^\pi \mathrm{ce}_{2n}(\eta, -h^2)\, \mathrm{d}\eta = (-1)^n \pi A_0^{(2n)}(h^2),$$

where $A_0^{(2n)}(h^2)$ is the first coefficient in the Fourier series expansion of $\mathrm{ce}_{2n}(\eta, h^2)$. It may be noted here that $A_0^{(2n)}(-h^2) = (-1)^n A_0^{(2n)}(h^2)$.

This example has been chosen to illustrate the theory because, while being smooth and physically reasonable, it allows us to express the coefficients $D_n(h)$ explicitly.

In punch problems, one of the quantities of interest is the normal component of stress under the punch, i.e., $\tau_{zz}(x, y, 0)$, where

$$\tau_{zz}(x, y, 0) = \frac{4}{\pi f(f^2 - y^2)^{1/2}} \int_0^\infty \cos\frac{2hx}{f} \sum_{n=0}^\infty D_n(h) \times \mathrm{ce}_{2n}\left(\cos^{-1}\left(\frac{y}{f}\right), -h^2\right) \frac{\mathrm{Fek}'_{2n}(0, -h^2)}{\mathrm{Fek}_{2n}(0, -h^2)}\, \mathrm{d}h, \tag{5.2}$$

and the total force exerted on the punch is given by

$$\int_{-\infty}^\infty \int_{-f}^f \tau_{zz}(x, y, 0)\, \mathrm{d}y\, \mathrm{d}x = 4f \int_0^\infty \int_0^{\pi/2} \tau_{zz}(x, f\cos\eta, 0) \sin\eta\, \mathrm{d}\eta\, \mathrm{d}x.$$

Evaluation of $D_n(h)$, $\tau_{zz}(x, y, 0)$, and the total force can only be done numerically, but in view of recent progress in the techniques of computing Mathieu functions, this is by no means an impossible task.

To illustrate this observation, we take the particular case where $l = 3f$, so that the substantial part of the punch profile is long compared with its width. This has the effect that the factor exp $(-2lh/f)$ tends to zero quite rapidly as h increase, so that it is only necessary to compute the $D_n(h)$ for small values of h.

Moreover, for such a profile, it is only necessary to consider small values of h, for the following reason: for $h = 0$, Mathieu functions reduce to trigonometric functions, namely

$$\text{ce}_0(\eta, 0) = 2^{-1/2}, \quad \text{ce}_{2n}(\eta, 0) = \cos 2n\eta, \quad (n \geqslant 1)$$

and for small values of h the Mathieu functions remain close to these approximations. Hence, for $n > 2$ and h small, the coefficient $A_0^{(2n)}$ is small compared with 1, so that $D_n(h)$ is itself small ([10], Sections 3.27 to 3.25).

Using the method described by Arscott et al. [2], $A_0^{(2n)}(h^2)$ have been computed for $n = 0$, 1, 2 and $h = 0(0.1)2$. From these the quantities

$$X_n = \int_0^\pi \text{ce}_{2n}(\eta, -h^2)\, \text{d}\eta = (-1)^n \pi A_0^{(2n)}(h^2) \tag{5.3}$$

have been computed for the same n, h, and are given in Table 1a of the Appendix. Relation (5.1) then gives the $D_n(h)$.

To evaluate the stress τ_{zz} immediately below the center of the punch (i.e., at $x = 0, y = 0$), as well as at the edge of the punch (i.e., at $x = 0$, $y = f$), we proceed as follows: let

$$V_n(h) = \frac{\text{Fek}'_{2n}(0, -h^2)}{\text{Fek}_{2n}(0, -h^2)} \tag{5.4}$$

and

$$U_n(\eta, h) = \frac{1}{\delta} D_n(h) V_n(h)\, \text{ce}_{2n}(\eta, -h^2)$$

$$= \frac{(-1)^n}{\delta} D_n(h) V_n(h)\, \text{ce}_{2n}\left(\frac{\pi}{2} - \eta, h^2\right),$$

so that $\eta = 0$ corresponds to the edge of the strip and $\eta = \pi/2$ corresponds to the center of the strip. Then, from (5.2),

$$\tau_{zz}(0, y, 0) \approx \frac{4\delta}{\pi f (f^2 - y^2)^{1/2}} \int_0^\infty U\left(\cos^{-1}\frac{y}{f}, h\right) \text{d}h$$

where

$$U(\eta, h) = U_0(\eta, h) + U_1(\eta, h) + U_2(\eta, h).$$

Finally we truncate the above integral since the integrand is small when $h > 2$, and evaluate

$$I(\eta) = \int_0^2 U(\eta, h)\, \text{d}h.$$

The quantity $V_n(h)$ is computed by a general technique developed by the second author (to be published), and is also tabulated in the Appendix (Table 2).

The normal component of stress under the punch can now be approximated by

$$\tau_{zz}(0, y, 0) \approx \frac{4\delta}{\pi f(f^2 - y^2)^{1/2}} I(\eta),$$

where

$$I\left(\frac{\pi}{2}\right) \approx -0.4818, \quad I(0) \approx 0.4130,$$

and furthermore τ_{zz} is negative everywhere under the punch. Let

$$J(x) = \int_0^2 \cos \frac{2hx}{f} U\left(\frac{\pi}{2}, h\right) dh,$$

then along the center line of the punch in the direction of the positive x-axis (and also for $x < 0$ by symmetry) we have the following approximation:

$$\tau_{zz}(x, 0, 0) \approx \frac{4\delta}{\pi f^2} J(x).$$

The following table gives values of $J(x)$ for $x/l = 0(0.1)1.2$. It may be noted that the profile's concavity remains unchanged for $x/l > 0.58$.

x/l	$J(x)$	x/l	$J(x)$
0	−0.481 82	0.7	−0.180 27
0.1	−0.451 99	0.8	−0.138 19
0.2	−0.439 79	0.9	−0.102 82
0.3	−0.393 82	1.0	−0.073 67
0.4	−0.339 44	1.1	−0.049 97
0.5	−0.282 91	1.2	−0.030 82
0.6	−0.228 91		

As it was pointed out earlier, in punch problems where one assumes complete contact, there will be stress singularities at the edge of the contact region. The function obtained above clearly exhibits the expected singularity at $y = \pm f$ (i.e., at the edge of the strip).

Example 2.

In this example we consider a punch whose profile varies transversely as well as longitudinally, i.e., $H(x, \eta)$ is dependent on η as well as x.

Let

$$H(x, \eta) = \delta \frac{l^2 \sin^2 \eta}{l^2 + x^2}$$

where l and δ are as in the previous example. Then, following the steps outlined above,

$$T(h, \eta) = \delta \int_0^\infty \cos\left(\frac{2hx}{f}\right) \frac{l^2 \sin^2 \eta}{(l^2 + x^2)} \, dx$$

$$= \frac{2l\delta}{f} \sin^2 \eta \exp\left(-\frac{2lh}{f}\right).$$

Furthermore $M_0(h) = (2l\delta/f) \exp(-2lh/f)$ and for $i = 1,2,3,4$, $M_i(h) = 2^i(l\delta/f)\exp(-2lh/f)$. Hence conditions (c.1) to (c.8) of Section 4 are satisfied. We have

$$D_n(h) = \frac{4l\delta}{\pi f} \exp\left(-\frac{2lh}{f}\right) \int_0^\pi \mathrm{ce}_{2n}(\eta, -h^2) \sin^2 \eta \, d\eta$$

$$= \frac{(-1)^n l\delta}{f} [2A_0^{(2n)}(h^2) + A_2^{(2n)}(h^2)] \exp\left(-\frac{2lh}{f}\right), \tag{5.5}$$

where $A_0^{(2n)}$ and $A_2^{(2n)}$ are the first two coefficients in the Fourier series expansion of $\mathrm{ce}_{2n}(\eta, h^2)$.

If, in place of $\sin^2 \eta \; (= \frac{1}{2} - \frac{1}{2}\cos 2\eta)$, the expression for the punch profile involved higher trigonometric terms in η (so that z, in terms of y, were given by a polynomial of degree higher than the second), then the effect would be to introduce further terms in (5.5), but only a finite number of these. Thus the computations involved would be of the same order of magnitude, since numerical construction of a Mathieu function generally produces all the significant coefficients $A_{2r}^{(2n)}$.

We let $l = 3f$ (as in Example 1) and compute the values of

$$Y_n(h) = \int_0^\pi \mathrm{ce}_{2n}(\eta, -h^2) \sin^2 \eta \, d\eta$$

$$= (-1)^n \frac{\pi}{4} [2A_0^{(2n)}(h^2) + A_2^{(2n)}(h^2)]. \tag{5.6}$$

(These quantities are also tabulated in the Appendix.) Thence $D_n(h)$ is computed using (5.5).

Using the table of values of $V_n(h)$ and the appropriate values of $D_n(h)$ and $\mathrm{ce}_{2n}(\eta, -h^2)$, we obtain the following two approximations for $I(\eta)$; namely at $\eta = \pi/2$ (i.e., at the center of the punch) and at $\eta = 0$ (i.e., at the edge of the strip):

$$I\left(\frac{\pi}{2}\right) \approx -1.2711, \quad I(0) \approx 0.7968.$$

We note that the stress at the center of the punch ($\eta = \pi/2$) is negative. This is in accordance with the assumption that $\tau_{zz} = -p$ where p is the normal pressure applied to the punch. Also since $I(0) > 0$ and τ_{zz} is a continuous function of η, then for some value of η in $(0, \pi/2)$, $\tau_{zz} = 0$. This shows that the above example does not represent a complete contact problem since contact is lost near the edge of the strip.

It is worth stressing that this conclusion does not represent a failure of the mathematical analysis, but shows that the physical assumptions on which the analysis is based are inconsistent; the physical problem as posed has no solution.

Essentially a punch of the postulated form cannot maintain complete contact. The mathematical analysis has succeeded insofar as it has shown up the inconsistencies in the physical formulation.

In order to determine what, in fact, happens when a punch of the given shape indents the region one has to determine the actual contact region which is not the entire punch surface. This might be done by the following method which is essentially an iterative procedure, suggested by Dr G.M.L. Gladwell.

First we find the pressure p in the strip, where $p(x, y) = -\tau_{zz}(x, y, 0)$, and the contour in the x,y-plane, on which $p = 0$. The region of the x,y-plane bounded by this contour, i.e., where $p \geqslant 0$, is then chosen to be the new contact region, say S_1. Next making use of the expression (see Section 1)

$$\Psi(x, y, z) = \frac{1}{2\pi} \int\int_S \frac{p(x', y')}{R} \, dx' \, dy',$$

we find a new $p(x, y)$ such that the prescribed displacement function $w(x, y)$, where $w(x, y) = \Psi(x, y, 0)$ for $x, y \in S_1$, satisfies

$$w(x, y) = \frac{1}{2\pi} \int\int_{S_1} \frac{p(x', y')}{R} \, dx' \, dy'.$$

If $p(x, y) > 0$ for all $x, y \in S_1$, then we increase the region S_1 and repeat the last step to obtain a new $p(x, y)$. On the other hand if $p(x, y) < 0$ at some point in S_1, then the region S_1 is decreased and again a new $p(x, y)$ is obtained.

The above steps are repeated until we find the region, S_∞, such that

$$\Psi(x, y, 0) = w(x, y) \quad \text{for } (x, y) \in S_\infty$$

(where $w(x, y)$ is prescribed),

$$p(x, y) > 0 \quad \text{in } S_\infty$$

and

$$p(x, y) = 0 \text{ on the boundary of } S_\infty.$$

This method has the disadvantage that it does not specify a precise algorithm for increasing the region S_1. Alternative approaches are indicated in the work of Fridman and Chernina [5] and of Kalker and Van Randen [8].

Example 3

In this example we examine, briefly, a punch whose profile $H(x, \eta)$ is a combination of the profiles in the previous two examples, namely let

$$H(x, \eta) = \frac{l^2}{l^2 + x^2} (\delta_1 + \delta_2 \sin^2\eta),$$

where l is defined in Example 1 and δ_1, δ_2 are two parameters such that the maximum depth of the punch is given by $\delta_1 + \delta_2$. Clearly the normal component of stress at each point, under the punch, is approximately equal to the sum of the corresponding values given in Examples 1 and 2, with the appropriate values of δ_1 and δ_2. Near the edge of the punch we have

$$\tau_{zz} \approx \frac{4}{\pi f(f^2 - y^2)^{1/2}} (-0.4130\delta_1 + 0.7968\delta_2).$$

Consequently, when $\delta_1/\delta_2 > 1.93$, the above expression will be negative and this will ensure complete contact between the base of the punch and the elastic medium.

6. The strip-crack problem

Since crack and punch problems, when considered as boundary-value problems, can be treated along similar lines, here we shall briefly discuss the crack problem corresponding to the strip-punch problem. A more detailed treatment can be found in [4].

It is assumed that a crack has developed inside an elastic medium which satisfies the properties stated in Section 1. We shall also assume that the crack is opened out symmetrically by equal normal pressures applied to its faces in the sense that if the Cartesian coordinate system is set up with the origin placed inside the crack, then the crack is opened out symmetrically with respect to each of the planes $x = 0$, $y = 0$ and $z = 0$.

In the strip-crack problem the crack occupies the infinite strip S defined by $z = 0$ and $|y| < f$. Due to the assumed symmetry we need only consider an elastic medium occupying an infinite half-space where the crack-face is the region S which is now on the boundary of the half-space.

For zero shear stress across the plane of the crack the corresponding boundary-value problem can be stated as follows. A harmonic function Ψ is to be found such that

(i) $\nabla^2\Psi = 0$ for $z > 0$,
(ii) $\Psi \to 0$ as $R \to \infty$ in $z \geqslant 0$,
(iii) $\Psi(x, y, 0) = 0$ for (x, y) outside S, i.e., the normal component of displacement is zero on the plane $z = 0$ outside the strip; this condition is due to the assumption that pressure is applied symmetrically.
(iv) $\tau_{zz}(x, y, 0) = \partial\Psi/\partial z|_{z=0} = -p(x, y)$, when $(x, y) \in S$.

$p(x, y)$ is some prescribed function which as before is assumed to be symmetric about $x = 0$ and $y = 0$.

In terms of elliptic cylinder coordinates, (x, ξ, η) with $x \in (-\infty, \infty)$, $\eta \in [0, \pi]$ and $\xi \geqslant 0$, the above problem can be restated as:

(i)′ Equation (3.1) of Section 3 holds for $x \in (-\infty, \infty)$, $\eta \in [0, \pi]$ and $\xi > 0$,
(ii)′ $\Psi \to 0$ as $|x| \to \infty$ or $\xi \to \infty$, for $\eta \in [0, \pi]$,
(iii)′ $\Psi = 0$ at $\eta = 0, \pi$ for $\xi \geqslant 0$ and $x \in (-\infty, \infty)$,

(iv)′ $\frac{1}{f \sin \eta}\left(\left.\frac{\partial \Psi}{\partial \xi}\right|_{\xi=0}\right) = -q(x, \eta)$, where $\eta \in (0, \pi)$, $x \in (-\infty, \infty)$ and $q(x, \eta) \overset{\text{d.}}{=} p(x, y)$.

In addition $q(x, \eta)$ is assumed to be symmetric about $x = 0$ and $\eta = \pi/2$.

Let $t(x, \eta) \overset{\text{d.}}{=} -f \sin \eta \; q(x, \eta)$, then through a set of steps, similar to those outlined in Section 3, we obtain the formal solution

$$\Psi = \int_0^\infty \cos \frac{2hx}{f} \sum_{n=0}^{\infty} K_n(h) \operatorname{se}_{2n+1}(\eta, -h^2) \operatorname{Gek}_{2n+1}(\xi, -h^2) \, \mathrm{d}h$$

where $\operatorname{se}_{2n+1}(\eta, -h^2)$ is given by (3.5c) and $\operatorname{Gek}_{2n+1}(\xi, -h^2)$ is the corresponding solution of the modified Mathieu equation which tends to zero as ξ tends to infinity ([10], Sections 11.12 and 11.42).

Similarly from (iv)′ (inverting the Fourier cosine transform and using the orthogonality of Mathieu functions of the first kind) the coefficients $K_n(h)$ are given by

$$K_n(h) = \frac{8}{\pi^2 f \operatorname{Gek}'_{2n+1}(0, -h^2)} \times \int_0^\pi \int_0^\infty t(x, \eta) \operatorname{se}_{2n+1}(\eta, -h^2) \cos \frac{2hx}{f} \, \mathrm{d}x \, \mathrm{d}\eta,$$

and the normal component of stress across the x,y-plane outside the crack is given by

$$\tau_{zz}(x, y, 0) = \frac{1}{f \sinh \xi}\left(\left.\frac{\partial \Psi}{\partial \eta}\right|_{\eta=0}\right)$$

where $y > f$ (i.e., $\xi > 0$).

7. Further developments

A similar study has been made of a punch whose shape is a parabola; this involves use of parabolical coordinates, and in place of the function $\cos(2hx/f)$ we have a modified Mathieu function. It is hoped to publish this in a subsequent paper.

8. Acknowledgement

The authors are grateful to Dr G.M.L. Gladwell for suggesting the procedure described in Example 2, and to the referee for drawing their attention to references [5] and [8], and other helpful suggestions.

Appendix

Tables 1a and 1b give the values of $X_n(h)$, $Y_n(h)$, as defined by (5.3), (5.6), and Table 2 gives the values of $V_n(h)$ as defined by (5.4).

Table 1a. $X_n(h)$

h	$n = 0$	$n = 1$	$n = 2$
0	2.221 441 469	0	0
0.1	2.221 427 585	−0.007 583 848	0.000 001 636
0.2	2.221 219 435	−0.031 407 379	0.000 026 180
0.3	2.220 319 667	−0.070 588 656	0.000 132 537
0.4	2.217 914 985	−0.125 120 518	0.000 418 888
0.5	2.212 927 644	−0.194 299 759	0.001 022 707
0.6	2.204 135 395	−0.276 739 705	0.002 120 799
0.7	2.190 380 122	−0.370 164 399	0.003 929 384
0.8	2.170 843 211	−0.471 377 772	0.006 704 208
0.9	2.145 303 564	−0.576 506 244	0.010 740 687
1	2.114 259 410	−0.681 499 244	0.016 373 995
1.1	2.078 839 010	−0.782 723 147	0.023 978 907
1.2	2.040 534 726	−0.877 420 182	0.033 969 063
1.3	2.000 883 691	−0.963 886 119	0.046 794 988
1.4	1.961 218 361	−1.041 375 181	0.062 939 827
1.5	1.922 541 951	−1.109 845 921	0.082 911 162
1.6	1.885 514 457	−1.169 672 890	0.107 226 584
1.7	1.850 503 951	−1.221 402 588	0.136 390 015
1.8	1.817 660 869	−1.265 582 147	0.170 855 381
1.9	1.786 898 172	−1.302 659 052	0.210 974 848
2	1.758 403 358	−1.332 940 592	0.256 931 295

Table 1b. $Y_n(h)$

h	$n = 0$	$n = 1$	$n = 2$
0	1.110 720 735	−0.785 398 164	0
0.1	1.107 937 039	−0.789 319 906	0.000 655 317
0.2	1.099 505 563	−0.801 018 988	0.002 631 096
0.3	1.085 203 326	−0.820 273 785	0.005 956 894
0.4	1.064 722 670	−0.846 641 808	0.010 682 201
0.5	1.037 775 886	−0.879 367 502	0.016 876 792
0.6	1.004 247 138	−0.917 295 911	0.024 631 213
0.7	0.964 370 315	−0.958 843 429	0.034 057 399
0.8	0.918 877 736	−1.002 079 405	0.045 289 377
0.9	0.869 042 367	−1.044 937 847	0.058 483 987
1	0.816 559 436	−1.085 509 201	0.073 821 469
1.1	0.763 288 377	−1.122 303 308	0.091 505 643
1.2	0.710 953 694	−1.154 381 690	0.111 763 259
1.3	0.660 918 525	−1.181 329 967	0.134 841 826
1.4	0.614 089 782	−1.203 118 703	0.161 004 883
1.5	0.570 942 006	−1.219 929 491	0.190 523 278
1.6	0.531 608 642	−1.232 004 224	0.223 660 509
1.7	0.495 990 460	−1.239 541 809	0.206 649 866
1.8	0.463 850 697	−1.242 643 044	0.301 661 153
1.9	0.434 885 726	−1.241 295 658	0.346 755 794
2	0.408 771 762	−1.235 392 165	0.395 831 869

Table 2. $V_n(h) = \dfrac{\mathrm{Fek}'_{2n}(0, -h^2)}{\mathrm{Fek}_{2n}(0, -h^2)}$

h	V_0	V_1	V_2
0.0	0	−2	−4
0.1	−0.409 35	−2.006 70	−4.002 72
0.2	−0.556 63	−2.026 06	−4.010 59
0.3	−0.690 87	−2.057 66	−4.023 98
0.4	−0.819 26	−2.102 21	−4.042 53
0.5	−0.943 30	−2.155 03	−4.066 20
0.6	−1.062 95	−2.220 10	−4.095 17
0.7	−1.177 76	−2.295 68	−4.128 99
0.8	−1.287 15	−2.381 52	−4.167 88
0.9	−1.390 72	−2.477 21	−4.211 59
1.0	−1.488 32	−2.582 18	−4.260 03
1.1	−1.579 98	−2.695 49	−4.313 32
1.2	−1.666 08	−2.816 19	−4.371 20
1.3	−1.747 12	−2.943 17	−4.433 41
1.4	−1.823 65	−3.075 31	−4.500 45
1.5	−1.896 20	−3.211 30	−4.571 85
1.6	−1.965 29	−3.350 31	−4.647 70
1.7	−2.031 45	−3.491 13	−4.728 13
1.8	−2.094 99	−3.632 80	−4.813 01
1.9	−2.156 29	−3.774 58	−4.902 68
2.0	−2.214 64	−3.915 58	−4.997 25

References

1. F.M. Arscott, *Periodic Differential Equations*, Macmillan Company (1964).
2. F.M. Arscott, R. Lacroix and W.T. Shymanski, A three-term recursion and the computation of Mathieu functions, *Proc. 8th Manitoba Conference on Numerical Mathematics and Computing* (1978) 107–115.
3. F.M. Arscott and A. Darai, Curvilinear co-ordinate systems in which the Helmholtz equation separates, *I.M.A. Journal of Applied Mathematics* 27 (1981) 33–70.
4. A. Darai, Applications of Higher Special Functions to Some Three-Dimensional Contact Problems in the Classical Theory of Elasticity, Ph.D. Thesis, University of Manitoba (1985).
5. V.M. Fridman and V.W. Chernina, The solution of a contact problem for a rigid body by an iteration method. (In Russian). *MEKH.TVERDOGO TELA AN SSSR1* (1967) 116–120.
6. G.M.L. Gladwell, *Contact Problems in the Classical Theory of Elasticity*, Sijthoff and Noordhoff (1980).
7. E.L. Ince, *Ordinary Differential Equations*, Longmans, Green and Co. (1927).
8. J.J. Kalker and Y. Van Randen, A minimum principle for frictionless elastic contact with application to non-Hertzian half-space contact problems, *J. Engg. Maths.* 6 (1972) 193–206.
9. A.I. Luré, *Three-Dimensional Problems of the Theory of Elasticity*, English translation, edited by J.R.M. Radok, New York, Interscience (1964).
10. N.W. McLachlan, *Theory and Application of Mathieu Functions*, Oxford University Press (1947).
11. R. Shail, Lamé polynomial solutions to some elliptic crack and punch problems, *Int. Journal of Engg. Sci.* 16 (1978) 551–563.